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$PSL(2,\mathbf{Z})$ の有限型不変量について

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0. Preliminaries

$PSL(2,\mathbf{Z})$ is the group of 2×2 matrices over $\mathbf{Z}$ with determinant 1 modulo $\pm E$. This group has the following generators

$$S=\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad T=\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

satisfying the relations

$$S^2=(TS)^3=E.$$

Any element of $PSL(2,\mathbf{Z})$ can be presented as follows by using S and T,

$$PSL(2,\mathbf{Z})\ni T^{b_1}ST^{b_2}S\cdots T^{b_l}S.$$

From now on, we use the following sequence of integers to indicate the element.

$$[\,b_1,b_2,\cdots,b_l\,]$$

Then we get the following relations by using this symbol.

$$[\,b_1,b_2,\cdots,b_i,0,b_{i+2},\cdots,b_l\,]=[\,b_1,b_2,\cdots,b_i+b_{i+2},\cdots,b_l\,]$$

$$[\,b_1,b_2,\cdots,b_i,1,1,1,b_{i+4},\cdots,b_l\,]=[\,b_1,b_2,\cdots,b_i,b_{i+4},\cdots,b_l\,]$$

It is known that two symbols present the same element in $PSL(2,\mathbf{Z})$ if and only if they can be transformed to each other by finite sequence of the above relations.

1. The definition of the finite type invariant of $PSL(2,\mathbf{Z})$

Let $\bar{\Gamma}$ denote the free abelian group generated by all the elements in $PSL(2,\mathbf{Z})$ and $\bar{\Gamma}_n$ denote the group spanned by the following set

$$\left\{ \sum_{c_{i_j}=\pm 1} (-1)^{\text{the number of } (-1)\text{'s in } \{c_{i_j}\}} \times [\, b_1, b_2, \cdots, b_l \,]_{c_{i_1}, c_{i_2}, \cdots, c_{i_n}} \right\},$$

where

$$\begin{aligned} &[\, b_1, b_2, \cdots, b_{i_1}, \cdots, b_{i_2}, \cdots, b_{i_n}, \cdots, b_l \,]_{c_{i_1}, c_{i_2}, \cdots, c_{i_n}} \\ &= [\, b_1, b_2, \cdots, b_{i_1} - c_{i_1} + 1, \cdots, b_{i_2} - c_{i_2} + 1, \cdots, b_{i_n} - c_{i_n} + 1, \cdots, b_l \,]. \end{aligned}$$

Note that if c_{i_j} is 1, then b_{i_j} does not change and that if c_{i_j} is -1, then b_{i_j} is changed to $b_{i_j} + 2$.

Now we define the finite type invariant of $PSL(2,\mathbf{Z})$ as following.

Definition. An additive map from $\bar{\Gamma}/\bar{\Gamma}_{n+1}$ to $\mathbf{Q}$ is called an invariant of type n.

Let $\sim_n$ (we call this n-equivalence) denote the equivalence relation defined by $\bar{\Gamma}_{n+1}$ in $\bar{\Gamma}$.

2. ON TYPE 0, 1 AND 2 INVARIANTS

Theorem 1.

$$\bar{\Gamma}/\bar{\Gamma}_1 = \mathbf{Z}\{[\ ], [\,0\,], [\,1\,], [\,0,1\,], [\,1,0\,], [\,1,1\,]\}.$$

Moreover, 0-equivalence class of $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ *is determined by its congruence class modulo 2.*

From now on, we restrict ourselves to the matrices 0-equivalent to the identity E and consider finite type invariants. Let $\bar{\Gamma}(2)$ be the span over $\mathbf{Z}$ of matrices 0-equivalent to the identity. We know that any element of $\bar{\Gamma}(2)$ can be presented as a sequence of even integers with even length, subject to the following relation

$$\begin{aligned} &[\, 2a_1, 2a_2, \cdots, 2a_i, 0, 2a_{i+2}, \cdots, 2a_{2m} \,] \\ &= [\, 2a_1, 2a_2, \cdots, 2(a_i + a_{i+2}), \cdots, 2a_{2m} \,]. \end{aligned}$$

By similar calculation, we have the following

Theorem 2.

$$\bar{\Gamma}(2)/\bar{\Gamma}(2)_2 = \mathbf{Z}\{[\ ], [\,0,2\,], [\,2,0\,]\}.$$

In fact, any element of $\bar{\Gamma}(2)$ *is 1-equivalent to*

$$(1-A)[\ ] + A_0\,[\,0,2\,] + A_1\,[\,2,0\,],$$

where

$$A = \sum_{i=1}^{2m} a_i, \quad A_0 = \sum_{i=1}^{m} a_{2i}, \quad A_1 = \sum_{i=1}^{m} a_{2i-1}.$$

Moreover, $1-A$*,* A_0*,* A_1 *are well-defined.*

If $[\,2a_1, 2a_2, \cdots, 2a_{2m}\,] = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$*, then*

$$A_0 = \sum_{i=1}^{\gamma/2} (-1)^{[(2i-1)\frac{\alpha}{\gamma}]}, \quad A_1 = \sum_{i=1}^{\gamma/2} (-1)^{[(2i-1)\frac{\delta}{\gamma}]}.$$

Where [ ] *denotes the greatest integer function.*

To prove the formulas, we use Tuler's result of the linking number of a 2-bridge link ([**2**]).

Corollary 2.1. *Any type 1 invariant is of the form*

$$c_1(1-A) + c_2A_0 + c_3A_1,$$

where c_i*'s are constants.*

Theorem 3.

$$\bar{\Gamma}(2)/\bar{\Gamma}(2)_3 = \mathbf{Z}\{[\ ], [\,0,2\,], [\,2,0\,], [\,2,2\,], [\,0,2,2,0\,], [\,0,4\,], [\,4,0\,]\}.$$

In fact, any element of $\bar{\Gamma}(2)$ *is 2-equivalent to*

$$\frac{(A-1)(A-2)}{2}[\ ] - A_0(A-2)\,[\,0,2\,] + A_1(A-2)\,[\,2,0\,]$$
$$+ \sum_{i=1}^{m}\sum_{j=i}^{m} a_{2i-1}a_{2j}\,[\,2,2\,] + \sum_{i=1}^{m-1}\sum_{j=i}^{m-1} a_{2i}a_{2j+1}\,[\,0,2,2,0\,]$$

$$+\frac{A_0(A_0-1)}{2}[\,0,4\,]+\frac{A_1(A_1-1)}{2}[\,4,0\,].$$

If $[\,2a_1, 2a_2, \cdots, 2a_{2m}\,] = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, *then*

$$\sum_{i=1}^{m}\sum_{j=i}^{m} a_{2i-1}a_{2j} = \sum_{i=1}^{(\alpha-1)/2}\sum_{j=i}^{(\alpha-1)/2}(-1)^{[(2i-1)\frac{\gamma}{\alpha}]+[2j\frac{\gamma}{\alpha}]},$$

$$\sum_{i=1}^{m-1}\sum_{j=i}^{m-1} a_{2i}a_{2j+1} = \sum_{i=1}^{(\alpha-1)/2}\sum_{j=i}^{(\alpha-1)/2}(-1)^{[(2i-1)\frac{\gamma}{\delta}]+[2j\frac{\gamma}{\delta}]}.$$

To prove the formulas, we use the result of the Casson knot invariant of a 2-bridge knot ([**1**]).

Corollary 3.1. *Any type 2 invariant is of the form*

$$d_1\frac{(A-1)(A-2)}{2} + d_2A_0(A-2) + d_3A_1(A-2)$$

$$+d_4\sum_{i=1}^{(\alpha-1)/2}\sum_{j=i}^{(\alpha-1)/2}(-1)^{[(2i-1)\frac{\gamma}{\alpha}]+[2j\frac{\gamma}{\alpha}]}+d_5\sum_{i=1}^{(\alpha-1)/2}\sum_{j=i}^{(\alpha-1)/2}(-1)^{[(2i-1)\frac{\gamma}{\delta}]+[2j\frac{\gamma}{\delta}]}$$

$$+d_6\frac{A_0(A_0-1)}{2} + d_7\frac{A_1(A_1-1)}{2},$$

where d_i*'s are constants.*

Detail will appear elsewhere.

References

[**1**] Y. Mizuma: *A formula for the Casson knot invariant of a* 2*-bridge knot,* to appear in J. Knot Theory Ramifications.

[**2**] R. Tuler: *On the linking number of a* 2*-bridge link,* Bull. London Math. Soc. **13** (1981), 540–544.